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Estimates for capacity and discrepancy of convex surfaces in sieve-like domains with an application to homogenization

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Abstract We consider the intersection of a convex surface Γ with a periodic perforation of \mathbb{R}^d , which looks like a sieve, given by $T_\varepsilon = \bigcup_{k \in \mathbb{Z}^d} \{\varepsilon k + a_\varepsilon T\}$ where T is a given compact set and $a_\varepsilon \ll \varepsilon$ is the size of the perforation in the ε -cell $(0, \varepsilon)^d \subset \mathbb{R}^d$. When ε tends to zero we establish uniform estimates for p -capacity, $1 < p < d$, of the set $\Gamma \cap T_\varepsilon$. Additionally, we prove that the intersections $\Gamma \cap \{\varepsilon k + a_\varepsilon T\}_k$ are uniformly distributed over Γ and give estimates for the discrepancy of the distribution. As an application we show that the thin obstacle problem with the obstacle defined on the intersection of Γ and the perforations, in a given bounded domain, is homogenizable when $p < 1 + \frac{d}{4}$. This result is new even for the classical Laplace operator.

Mathematics Subject Classification 35R35 · 35B27 · 32U15 · 11K06

1 Introduction

In this paper we study the properties of the intersection of a convex surface Γ with a periodic perforation of \mathbb{R}^d given by $T_\varepsilon = \bigcup_{k \in \mathbb{Z}^d} \{\varepsilon k + a_\varepsilon T\}$, where T is a given compact set and a_ε is the size of the perforation in the ε -cell. Our primary interest is to obtain good control of p -capacity $1 < p < d$ and discrepancy of distributions of the components of the intersection $\Gamma \cap T_\varepsilon$ in terms of ε when the size of perforations tends to zero. As an application of our analysis we get that the thin obstacle problem in periodically perforated domain $\Omega \subset \mathbb{R}^d$

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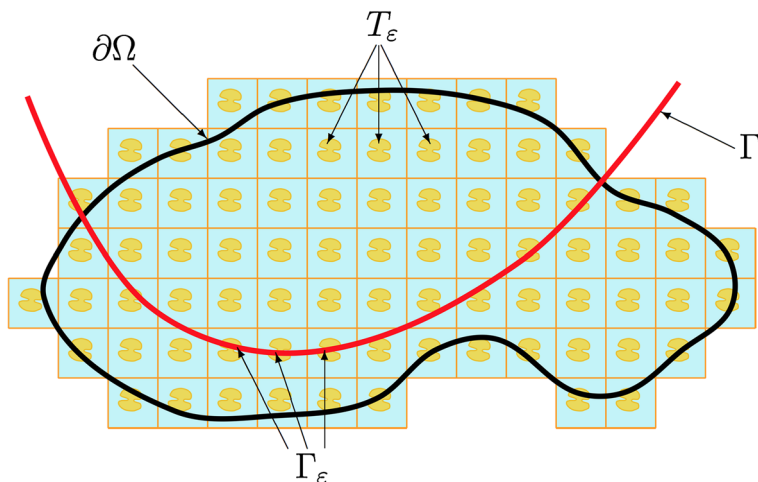


Fig. 1 The sieve-like configuration with convex Γ

with given strictly convex and C^2 smooth surface as the obstacle and p -Laplacian as the governing partial differential equation is homogenizable provided that $p < 1 + \frac{d}{4}$. Moreover, the limit problem admits a variational formulation with one extra term involving the mean capacity, see Theorem 3. The configuration of Γ , Γ_ε , T_ε and Ω is illustrated in Fig. 1.

This result is new even for the classical case $p = 2$ corresponding to the Laplace operator. Another novelty is contained in the proof of Theorem 2 where we use a version of the method of quasi-uniform continuity developed in [4].

1.1 Statement of the problem

Let

$$T_\varepsilon = \bigcup_{k \in \mathbb{Z}^d} \{\varepsilon k + a_\varepsilon T\},$$

and let

$$\Gamma_\varepsilon = \Gamma \cap T_\varepsilon.$$

We assume that Γ is a strictly convex surface in \mathbb{R}^d that locally admits the representation

$$\{(x', g(x')) : x' \in Q'\}, \quad (1)$$

where $Q' \subset \mathbb{R}^{d-1}$ is a cube. For example, Γ may be a compact convex surface, or may be defined globally as a graph of a convex function.

Without loss of generality we assume that $x_d = g(x')$ because the interchanging of coordinates preserves the structure of the periodic lattice in the definition of T_ε . We will also study homogenization of the thin obstacle problem for the p -Laplacian with an obstacle defined on Γ_ε . Our goal is to determine the asymptotic behaviour, as $\varepsilon \rightarrow 0$, of the problem

$$\min \left\{ \int_{\Omega} |\nabla v|^p dx + \int_{\Omega} h v dx : v \in W_0^{1,p}(\Omega) \text{ and } v \geq \phi \text{ on } \Gamma_\varepsilon \right\}, \quad (2)$$

for given $h \in L^q(\Omega)$, $1/p + 1/q = 1$ and $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

We make the following assumptions on Ω , T , Γ , d and p :

(A₁) $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain.

(A₂) The compact set T from which the holes are constructed must be sufficiently regular in order for the mapping

$$t \mapsto \text{cap}(\{\Gamma + te\} \cap T)$$

to be continuous, where e is any unit vector. This is satisfied if, for example, T has Lipschitz boundary.

(A₃) The size of the holes is

$$a_\varepsilon = \varepsilon^{d/(d-p+1)}.$$

This is the critical size that gives rise to an interesting effective equation for (2).

(A₄) The exponent p in (2) is in the range

$$1 < p < \frac{d+4}{4}.$$

This is to ensure that the holes are large enough that we are able to effectively estimate the intersections between the surface Γ and the holes T_ε , of size a_ε . See the discussion following the estimate (15). In particular, if $p = 2$ then $d > 4$.

These are the assumptions required for using the framework from [4], though the (A₄) is stricter here.

1.2 Main results

The following theorems contain the main results of the present paper.

Theorem 1 Suppose Γ is a C^2 convex surface. Let $I_\varepsilon \subset [0, 1)$ be an interval, let $Q' \subset \mathbb{R}^{d-1}$ be a cube and let

$$A_\varepsilon = \# \left\{ k' \in \mathbb{Z}^{n-1} \cap \varepsilon^{-1} Q' : \frac{g(\varepsilon k')}{\varepsilon} \in I_\varepsilon \pmod{1} \right\}.$$

Then

$$\left| \frac{A_\varepsilon}{N_\varepsilon} - |I_\varepsilon| \right| = O(\varepsilon^{\frac{1}{3}}),$$

where $N_\varepsilon = \#\{k' \in \mathbb{Z}^{d-1} \cap \varepsilon^{-1} Q'\}$.

Next we establish an important approximation result. We use the notation $T_\varepsilon^k = \varepsilon k + a_\varepsilon T$ and $\Gamma_\varepsilon^k = \Gamma \cap T_\varepsilon^k$.

Theorem 2 Suppose Γ is a C^2 convex surface and P_x a support plane of Γ at the point $x \in \Gamma$. Then

1° the p -capacity of $P_x^k = P_x \cap T_\varepsilon^k$ approximates $\text{cap}_p(\Gamma_\varepsilon^k)$ as follows

$$\text{cap}_p(\Gamma_\varepsilon^k) = \text{cap}_p(P_x^k \cap \{a_\varepsilon T + \varepsilon k\}) + o(a_\varepsilon^{d-p}), \quad (3)$$

where $x \in \Gamma_\varepsilon^k$.

2° Furthermore, if P_1 and P_2 are two planes that intersect $\{a_\varepsilon T + \varepsilon k\}$ at a point x , with normals v_1, v_2 satisfying $|v_1 - v_2| \leq \delta$ for some small $\delta > 0$, then

$$|\operatorname{cap}_p(P_1 \cap \{a_\varepsilon T + \varepsilon k\}) - \operatorname{cap}_p(P_2 \cap \{a_\varepsilon T + \varepsilon k\})| \leq c_\delta a_\varepsilon^{d-p}, \quad (4)$$

where $\lim_{\delta \rightarrow 0} c_\delta = 0$.

As an application of Theorems 1, 2 we have

Theorem 3 Let u_ε be the solution of (2). Then $u_\varepsilon \rightarrow u$ in $W_0^{1,p}(\Omega)$ as $\varepsilon \rightarrow 0$, where u is the solution to

$$\min \left\{ \int_\Omega |\nabla v|^p dx + \int_{\Gamma \cap \Omega} |(\phi - v)_+|^p \operatorname{cap}_{p,v(x)}(T) dH^{d-1} + \int_\Omega f v dx : v \in W_0^{1,p}(\Omega) \right\}. \quad (5)$$

In (5), $v(x)$ is the normal of Γ at $x \in \Gamma$ and $\operatorname{cap}_{p,v(x)}(T)$ is the mean p -capacity of T with respect to the hyperplane $P_{v(x)} = \{y \in \mathbb{R}^d : v(x) \cdot y = 0\}$, given by

$$\operatorname{cap}_{p,v(x)}(T) = \int_{-\infty}^{\infty} \operatorname{cap}_p(T \cap \{P_{v(x)} + tv(x)\}) dt, \quad (6)$$

where $\operatorname{cap}_p(E)$ denotes p -capacity of E with respect to \mathbb{R}^d .

Theorem 3 was proved by the authors in [4] under the assumption that Γ is a hyper plane, which was in turn a generalization of the paper [5]. In a larger context, Theorem 3 contributes to the theory of homogenization in non-periodic perforated domains, in that the support of the obstacle, Γ_ε , is not periodic. Another class of well-studied non-periodic perforated domains, not including that of the present paper, is the random stationary ergodic domains introduced in [1]. In the case of stationary ergodic domains the perforations are situated on lattice points, which is not the case for the set Γ_ε . The perforations, i.e. the components of Γ_ε , have desultory (though deterministic by definition) distribution. For the periodic setting [2] is a standard reference.

The proof of Theorem 3 has two fundamental ingredients. First the structure of the set Γ_ε is analysed using tools from the theory of uniform distribution, Theorem 1. We prove essentially that the components of Γ_ε are uniformly distributed over Γ with a good bound on the discrepancy. This is achieved by studying the distribution of the sequence

$$\{\varepsilon^{-1} g(\varepsilon k')\}_{k'}, \quad (7)$$

for g defined by (1) and $\varepsilon k' \in Q'$. Second, we construct a family of well-behaved correctors based on the result of Theorem 2.

The major difficulty that arises when Γ is a more general surface than a hyperplane is to estimate the discrepancy of the distribution of (the components of) Γ_ε over Γ , which is achieved through studying the discrepancy of $\{\varepsilon^{-1} g(\varepsilon k')\}_{k'}$. For a definition of discrepancy, see Sect. 2. In the framework of uniform convexity we can apply a theorem of Erdős and Koksma which gives good control of the discrepancy.

2 Discrepancy and the Erdős–Koksma theorem

In this section we formulate a general result for the uniform distribution of a sequence and derive a decay estimate for the corresponding discrepancy.

Definition 1 The discrepancy of the first N elements of a sequence $\{s_j\}_{j=1}^\infty$ is given by

$$D_N = \sup_{I \subset (0,1]} \left| \frac{A_N}{N} - |I| \right|,$$

where I is an interval, $|I|$ is the length of I and A_N is the number of $1 \leq j \leq N$ for which $s_j \in I \pmod{1}$.

We first recall the Erdős–Turán inequality, see Theorem 2.5 in [7], for the discrepancy of the sequence $\{s_j\}_{j=1}^\infty$

$$D_N \leq \frac{1}{n} + \frac{1}{N} \sum_{k=1}^n \frac{1}{k} \left| \sum_{j=1}^N e^{2\pi i f(j)k} \right| \quad (8)$$

where n is a parameter to be chosen so that the right hand side has optimal decay as $N \rightarrow \infty$. Observe that s_j is the j -th element of the sequence which in our case is $s_j = f(j)$ for a given function f and $N = \lceil \frac{1}{\varepsilon} \rceil$.

We employ the following estimate of Erdős and Koksma ([7], Theorem 2.7) in order to estimate the second sum in (8): let $a, b \in \mathbb{N}$ such that $0 < a < b$ then one has the estimate

$$\left| \sum_{j=1}^N e^{2\pi i f(j)k} \right| \leq (|F'_k(b) - F'_k(a)| + 2) \left(3 + \frac{1}{\sqrt{\rho}} \right) \quad (9)$$

where $F_k(t) = kf(t)$ and $F''_k(t) \geq \rho > 0$ for some positive number ρ . In order to apply this result to our problem we first need to reduce the dimension of (7) to one. To do so let us assume that the obstacle Γ is given as the graph of a function $x_d = g(x')$ where g is strictly convex C^2 function such that

$$c_0 \delta_{\alpha, \beta} \leq D_{x_\alpha x_\beta} g(x') \leq C_0 \delta_{\alpha, \beta}, \quad 1 \leq \alpha, \beta \leq d-1 \quad (10)$$

for some positive constants $c_0 < C_0$.

Next we rescale the ε -cells and consider the normalised problem in the unit cube $[0, 1]^d$. The resulting function is $f(j) = \frac{g(\varepsilon j)}{\varepsilon}$, $j \in \mathbb{Z}^{d-1}$.

If $d = 2$ then we can directly apply (9) to the scaled function f above. Otherwise for $d > 2$ we need an estimate for the multidimensional discrepancy in terms of D_N introduced in Definition 1, a similar idea was used in [4] for the linear obstacle. Suppose for a moment that this is indeed the case. Then we can take $F_k(t) = kf(t)$ in (9) and noting

$$D_{x_\alpha} f(x') = k D_\alpha g(\varepsilon x'), \quad D_{x_\alpha}^2 f(x') = k \varepsilon D_\alpha^2 g(\varepsilon x') \geq k \varepsilon c_0, \quad 1 \leq \alpha \leq d-1 \quad (11)$$

one can proceed as follows

$$\begin{aligned} \left| \sum_{j=1}^N e^{2\pi i f(j)k} \right| &\leq (|k D_{x_\alpha} g(\varepsilon N) - k D_\alpha g(\varepsilon)| + 2) \left(3 + \frac{1}{\sqrt{k \varepsilon c_0}} \right) \\ &\leq (k \varepsilon C_0 (N-1) + 2) \left(3 + \frac{1}{\sqrt{k \varepsilon c_0}} \right) \\ &\leq k \left(\varepsilon C_0 (N-1) + \frac{2}{k} \right) \left(3 + \frac{1}{\sqrt{k \varepsilon c_0}} \right) \end{aligned}$$

$$\begin{aligned}
&\leq k \left(\varepsilon C_0(N-1) + \frac{2}{k} \right) \left(3 + \sqrt{\frac{N}{kC_0}} \right) \\
&\leq \lambda k \left(1 + \sqrt{\frac{N}{k}} \right)
\end{aligned}$$

for some tame constant $\lambda > 0$ independent of ε, k . Plugging this into (8) yields

$$\begin{aligned}
D_N &\leq \frac{1}{n} + \frac{\lambda}{N} \sum_{k=1}^n \left(1 + \sqrt{\frac{N}{k}} \right) \\
&= \frac{1}{n} + \frac{\lambda n}{N} + \frac{\lambda}{\sqrt{N}} \sum_{k=1}^n \frac{1}{\sqrt{k}} \\
&\leq \frac{1}{n} + \bar{\lambda} \sqrt{\frac{n}{N}} \left(1 + \sqrt{\frac{n}{N}} \right)
\end{aligned}$$

for another tame constant $\bar{\lambda} > 0$. Now to get the optimal decay rate we choose $\frac{1}{n} = \sqrt{\frac{n}{N}}$ which yields $N = n^3$ and hence

$$n = N^{\frac{1}{3}} \approx \frac{1}{\varepsilon^{\frac{1}{3}}}$$

and we arrive at the estimate

$$D_N = O(\varepsilon^{\frac{1}{3}}). \quad (12)$$

2.1 Proof of Theorem 1

Proof Suppose Q' is a cube of size r . Then there is a cube $Q'' \subset \mathbb{R}^{d-2}$ such that $Q' = [\alpha, \beta] \times Q'', \beta - \alpha = r$. We may rewrite A_ε as

$$A_\varepsilon = \sum_{k'' \in \varepsilon^{-1}Q'' \cap \mathbb{Z}^{d-2}} \# \{k_1 \in \mathbb{Z}: a \leq k_1 \leq b \text{ and } \varepsilon^{-1}g(\varepsilon k_1 + \varepsilon k'') \in I_\varepsilon \pmod{1}\},$$

where $(k_1, k'') = k'$, a, b are the integer parts of $\varepsilon^{-1}\alpha$ and $\varepsilon^{-1}\beta$ respectively and $|(b-a) - \varepsilon^{-1}r| \leq 1$. We also note that $N_\varepsilon = (\varepsilon^{-1}r)^{d-1} + O(\varepsilon^{-1}r)^{d-2}$. Consider

$$A_\varepsilon^1(k'') = \# \{k_1 \in \mathbb{Z}: a \leq k_1 \leq b \text{ and } \varepsilon^{-1}g(\varepsilon k_1 + \varepsilon k'') \in I_\varepsilon \pmod{1}\}.$$

Then we have

$$\frac{A_\varepsilon}{N_\varepsilon} - |I_\varepsilon| = \frac{1}{(\varepsilon^{-1}r)^{d-2}} \sum_{k'' \in \varepsilon^{-1}Q'' \cap \mathbb{Z}^{d-2}} \frac{A_\varepsilon^1(k'')}{(\varepsilon^{-1}r)} - |I_\varepsilon|. \quad (13)$$

For each k'' the function $h: s \rightarrow \varepsilon^{-1}g(\varepsilon s + \varepsilon k'')$ satisfies $|h'(s)| \leq C_1$ and $h''(s) \geq \rho\varepsilon$ for $a \leq s \leq b$. Thus we may apply the Erdős-Koksma Theorem as described above and conclude that

$$\left| \frac{A_\varepsilon^1(k'')}{(\varepsilon^{-1}r)} - |I_\varepsilon| \right| \leq C\varepsilon^{\frac{1}{3}}.$$

It follows that the modulus of the left hand side of (13) is bounded by $C\varepsilon^{\frac{1}{3}}$, proving the theorem. \square

3 Correctors

The purpose of this section is to construct a sequence of correctors that satisfy the hypotheses given below. Once we have established the existence of these correctors, the proof of the Theorem 3 is identical to the planar case treated in [4].

H1 $0 \leq w_\varepsilon \leq 1$ in \mathbb{R}^d , $w_\varepsilon = 1$ on Γ_ε and $w_\varepsilon \rightarrow 0$ in $W_{\text{loc}}^{1,p}(\mathbb{R}^d)$,

H2 $\int_\Omega |\nabla w_\varepsilon|^p dx \rightarrow \int_\Gamma f(x) \text{cap}_{p,v_x} d\mathcal{H}^{s-\infty}$, for any $f \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$,

H3 (weak continuity) for any $\phi_\varepsilon \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that

$$\begin{cases} \sup_{\varepsilon>0} \|\phi_\varepsilon\|_{L^\infty(\Omega)} < \infty, \\ \phi_\varepsilon = 0 \text{ on } \Gamma_\varepsilon \text{ and } \phi_\varepsilon \rightharpoonup \phi \in W_0^{1,p}(\Omega), \end{cases}$$

we have

$$\langle -\Delta_p w_\varepsilon, \phi_\varepsilon \rangle \rightarrow \langle \mu, \phi \rangle$$

with

$$d\mu(x) = \text{cap}_{p,v(x)} d\mathcal{H}^{s-\infty} \llcorner \Gamma, \quad (14)$$

where $\text{cap}_{p,v(x)}$ is given by (6) and $\mathcal{H}^s \llcorner \Gamma$ is the restriction of s -dimensional Hausdorff measure on Γ .

Setting $\Gamma_\varepsilon^k := \Gamma \cap \{a_\varepsilon T + \varepsilon k\} \neq \emptyset$, we define w_ε^k by

$$\begin{aligned} \Delta_p w_\varepsilon^k &= 0 && \text{in } B_{\varepsilon/2}(\varepsilon k) \setminus \Gamma_\varepsilon^k, \\ w_\varepsilon^k &= 0 && \text{on } \partial B_{\varepsilon/2}(\varepsilon k), \\ w_\varepsilon^k &= 1 && \text{on } \Gamma_\varepsilon^k. \end{aligned}$$

Then it follows from the definition of cap_p [3] that

$$\int_{B_{\varepsilon/2}(\varepsilon k)} |\nabla w_\varepsilon^k|^p dx = \text{cap}_p(\Gamma_\varepsilon^k) + o(a_\varepsilon^{d-p}).$$

Indeed, we have

$$\begin{aligned} \text{cap}_p(\Gamma_\varepsilon^k, B_{\varepsilon/2}(\varepsilon k)) &= \inf \left\{ \int_{B_{\varepsilon/2}} |\nabla w|^p : w \in W_0^{1,p}(B_{\varepsilon/2}(\varepsilon k)) \text{ and } w = 1 \text{ on } \Gamma_\varepsilon^k \right\} \\ &= a_\varepsilon^{d-p} \inf \left\{ \int_{B_{\varepsilon/2a_\varepsilon}} |\nabla w|^p : w \in W_0^{1,p}(B_{\varepsilon/2a_\varepsilon}) \text{ and } w = 1 \text{ on } \frac{1}{a_\varepsilon} \Gamma_\varepsilon^k \right\} \\ &= a_\varepsilon^{d-p} \left(\text{cap}_p \left(\frac{1}{a_\varepsilon} \Gamma_\varepsilon^k \right) + o(1) \right) \\ &= \text{cap}_p(\Gamma_\varepsilon^k) + o(a_\varepsilon^{d-p}). \end{aligned}$$

Note that $\text{cap}_p(\Gamma_\varepsilon^k) = O(a_\varepsilon^{d-p})$ since $\Gamma_\varepsilon^k = \Gamma \cap \{\varepsilon k + a_\varepsilon T\}$ and $\text{cap}_p(tE) = t^{d-p} \text{cap}_p(E)$ if $t \in \mathbb{R}_+$ and $E \subset \mathbb{R}^d$. If Q' is a cube in \mathbb{R}^{d-1} , the components of $\Gamma_\varepsilon \cap Q' \times \mathbb{R}$ are of the form $\Gamma_\varepsilon^k = \Gamma \cap \{(\varepsilon k', \varepsilon k_d) + a_\varepsilon T\}$ for $\varepsilon k' \in Q'$. In particular, $\Gamma_\varepsilon^k \neq \emptyset$ if and only if

$\varepsilon^{-1}g(\varepsilon k') \in I_\varepsilon \pmod{1}$ where $|I_\varepsilon| = O(a_\varepsilon/\varepsilon)$. Thus Theorem 1 tells us that the number of components of $\Gamma_\varepsilon \cap Q' \times \mathbb{R}$ equals $A_\varepsilon = |I_\varepsilon|N_\varepsilon + N_\varepsilon O(\varepsilon^{\frac{1}{3}})$, or explicitly

$$\left| \frac{\frac{A_\varepsilon}{N_\varepsilon}}{\frac{a_\varepsilon}{\varepsilon}} - 1 \right| = \frac{O(\varepsilon^{\frac{1}{3}})}{\frac{a_\varepsilon}{\varepsilon}}. \quad (15)$$

Here we need to have $\varepsilon^{1/3} = o(|I_\varepsilon|)$, which is equivalent to (A₄). Since

$$\int_{B_{\varepsilon/2}(\varepsilon k)} |\nabla w_\varepsilon^k|^p dx = \text{cap}_p(\Gamma_\varepsilon^k) + o(a_\varepsilon^{d-p}),$$

we get

$$\int_{\mathbb{R} \times Q'} |\nabla w_\varepsilon|^p dx \leq C(|I_\varepsilon|N_\varepsilon \text{cap}_p(\Gamma_\varepsilon^k)) \leq C \frac{a_\varepsilon}{\varepsilon} \varepsilon^{1-d} |Q'| a_\varepsilon^{n-p} = C|Q'|.$$

Thus $\int_K |\nabla w_\varepsilon|^p$ is uniformly bounded on compact sets K . Since $w_\varepsilon(x) \rightarrow 0$ pointwise for $x \notin \Gamma$, **H1** follows.

When verifying **H2** and **H3** we will only prove that

$$\lim_{\varepsilon \rightarrow 0} \int_Q |\nabla w_\varepsilon|^p dx = \int_{\Gamma \cap Q} c_{\nu(x)} d\mathcal{H}^{d-1}(x), \quad \text{for all cubes } Q \subset \mathbb{R}^d. \quad (16)$$

Once this has been established the rest of the proof is identical to that given in [4].

4 Proof of Theorem 2

Proof **1°** Set $R_\varepsilon = \frac{\varepsilon}{2a_\varepsilon} \rightarrow \infty$, then after scaling we have to prove that

$$\int_{B_{R_\varepsilon}} |\nabla v_1|^p - \int_{B_{R_\varepsilon}} |\nabla v_2|^p = o(1) \quad (17)$$

uniformly in ε where

$$\begin{aligned} \Delta_p v_i &= 0 && \text{in } B_{R_\varepsilon} \setminus S_i, \\ v_i &= 0 && \text{on } \partial B_{R_\varepsilon}, \\ v_i &= 1 && \text{on } S_i. \end{aligned}$$

and $S_1 = \frac{1}{a_\varepsilon} \Gamma_\varepsilon^k$, $S_2 = \frac{1}{a_\varepsilon} P_X$.

We approximate v_i in the domain $B_{R_\varepsilon} \setminus D_i^t$ with D_i^t being a bounded domain with smooth boundary and $D_i^t \rightarrow S_i$ as $t \rightarrow 0$ in Hausdorff distance. Consider

$$\begin{aligned} \Delta_p v_i^t &= 0 && \text{in } B_{R_\varepsilon} \setminus D_i^t, \\ v_i^t &= 0 && \text{on } \partial B_{R_\varepsilon}, \\ v_i^t &= 1 && \text{on } \partial D_i^t. \end{aligned}$$

Observe that $\int_{B_{R_\varepsilon} \setminus D_i^t} |\nabla v_i^t|^p$, $i = 1, 2$ remain bounded as $t \rightarrow 0$ thanks to Caccioppoli's inequality. Indeed, $w = (1 - v_i^t)\eta \in W_0^{1,p}(B_5 \setminus D_i^t)$ where $\eta \in C_0^\infty(B_5)$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ in B_3 . Using w as a test function we conclude that

$$\int_{B_5 \setminus D_i^t} |\nabla v_i^t|^p \eta = \int_{B_5 \setminus D_i^t} |\nabla v_i^t|^{p-2} \nabla v_i^t \nabla \eta (1 - v_i^t).$$

Since $\eta \equiv 1$ in B_3 then applying Hölder inequality we infer that $\int_{B_3 \setminus D_1^t} |\nabla v_i^t|^p \leq C \int_{B_3} (1 - v_i^t)^p$. In $B_{R_\varepsilon} \setminus B_2$ the L^p we compare $W(x) = |x/2|^{\frac{p-d}{p-1}}$ with v_i . Note that our assumption A_4 implies that $p < d$. Moreover, since W is p -harmonic in $B_{R_\varepsilon} \setminus B_2$ then the comparison principle yields $v_i \leq W$ in $B_{R_\varepsilon} \setminus B_2$. From the proof of Caccioppoli's inequality above choosing non-negative $\eta \in C^\infty(\mathbb{R}^d)$ such that $\eta \equiv 0$ in B_2 , $\frac{1}{2} \leq \eta \leq 1$ in $B_{R_\varepsilon} \setminus B_3$, and $\eta = 1$ in $\mathbb{R}^d \setminus B_{R_\varepsilon}$ and using $\eta v_i \in W_0^{1,p}(B_{R_\varepsilon} \setminus B_2)$ as a test function we infer

$$\int_{B_{R_\varepsilon} \setminus B_3} |\nabla v_i|^p \leq \frac{C}{R_\varepsilon^p} \int_{B_{R_\varepsilon} \setminus B_2} v_i^p \leq \frac{C}{R_\varepsilon^{\frac{1}{p-1}}} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

where the last bound follows from the estimate $v_i \leq W$. Combining these estimates we infer

$$\|v_i^t\|_{W^{1,p}(B_{R_\varepsilon})} \leq K, \quad i = 1, 2 \quad (18)$$

for some tame constant K independent of t and ε . Thus, by construction $v_i^t \rightharpoonup v_i$ weakly in $W_0^{1,p}(B_{R_\varepsilon})$.

Let $\psi \in C^\infty(\mathbb{R}^d)$ such that $\text{supp } \psi \supset D_1^t \cup D_2^t$ and $\psi \equiv 1$ in $\mathbb{R}^d \setminus B_2$. Then the function $\psi(v_1^t - v_2^t) \in W_0^{1,p}(B_{R_\varepsilon})$ and it vanishes on $\text{supp } \psi \supset D_1^t \cup D_2^t$. Thus we have

$$\begin{aligned} & \int_{B_{R_\varepsilon}} (\nabla v_1^t |\nabla v_1^t|^{p-2} - \nabla v_2^t |\nabla v_2^t|^{p-2}) (\nabla v_1^t - \nabla v_2^t) \psi \\ &= - \int_{B_{R_\varepsilon}} (\nabla v_1^t |\nabla v_1^t|^{p-2} - \nabla v_2^t |\nabla v_2^t|^{p-2}) (v_1^t - v_2^t) \nabla \psi \end{aligned}$$

Note that $v_1^t - v_2^t = 0$ on $D_1^t \cap D_2^t$. Choosing a sequence ψ_n such that $1 - \psi_n$ converges to the characteristic function $\chi_{D_1^t \cup D_2^t}$ of the set $D_1^t \cup D_2^t$ we conclude

$$\int_{B_{R_\varepsilon}} (\nabla v_1^t |\nabla v_1^t|^{p-2} - \nabla v_2^t |\nabla v_2^t|^{p-2}) (\nabla v_1^t - \nabla v_2^t) = J_1 + J_2 \quad (19)$$

where

$$\begin{aligned} J_1 &= \int_{\partial D_1^t} (1 - v_2^t) [\partial_\nu v_1^t |\nabla v_1^t|^{p-2} - \partial_\nu v_2^t |\nabla v_2^t|^{p-2}], \\ J_2 &= \int_{\partial D_2^t} (v_1^t - 1) [\partial_\nu v_1^t |\nabla v_1^t|^{p-2} - \partial_\nu v_2^t |\nabla v_2^t|^{p-2}]. \end{aligned}$$

Notice that on ∂D_i^t we have that $\nu = -\frac{\nabla \psi_m}{|\nabla \psi_m|}$ is the unit normal pointing inside D_i^t . We denote $n = -\nu$ and then we have that

$$\begin{aligned} - \int_{\partial D_1^t} (1 - v_2^t) \partial_\nu v_2^t |\nabla v_2^t|^{p-2} &= \int_{\partial D_1^t} (1 - v_2^t) \partial_n v_2^t |\nabla v_2^t|^{p-2} \\ &= \int_{\partial(D_1^t \cap D_2^t)} (1 - v_2^t) \partial_n v_2^t |\nabla v_2^t|^{p-2} \\ &= \int_{D_1^t \setminus D_2^t} \text{div}((1 - v_2^t) \nabla v_2^t |\nabla v_2^t|^{p-2}) \\ &= - \int_{D_1^t \setminus D_2^t} |\nabla v_2^t|^p, \end{aligned}$$

and similarly

$$\int_{\partial D_2^t} (v_1^t - 1) \partial_\nu v_1^t |\nabla v_1^t|^{p-2} = - \int_{D_2^t \setminus D_1^t} |\nabla v_1^t|^p.$$

Setting

$$I = \int_{B_{R_\varepsilon}} (\nabla v_1^t |\nabla v_1^t|^{p-2} - \nabla v_2^t |\nabla v_2^t|^{p-2}) (\nabla v_1^t - \nabla v_2^t) \quad (20)$$

and returning to (19) we infer

$$\begin{aligned} I &= - \int_{D_1^t \setminus D_2^t} |\nabla v_2^t|^p - \int_{D_2^t \setminus D_1^t} |\nabla v_1^t|^p + \int_{\partial D_1^t} (1 - v_2^t) \partial_\nu v_1^t |\nabla v_1^t|^{p-2} \\ &\quad - \int_{\partial D_2^t} (v_1^t - 1) \partial_\nu v_2^t |\nabla v_2^t|^{p-2} \\ &\leq \int_{\partial D_1^t} (1 - v_2^t) \partial_\nu v_1^t |\nabla v_1^t|^{p-2} - \int_{\partial D_2^t} (v_1^t - 1) \partial_\nu v_2^t |\nabla v_2^t|^{p-2} \\ &\leq \sup_{D_1^t} (1 - v_2^t) \int_{\partial D_1^t} |\partial_\nu v_1^t| |\nabla v_1^t|^{p-2} + \sup_{D_2^t} (1 - v_1^t) \int_{\partial D_2^t} |\partial_\nu v_2^t| |\nabla v_2^t|^{p-2}. \end{aligned}$$

But on ∂D_i^t we have $\partial_\nu v_i^t \geq 0$ (v_i points inside D_i^t) because v_i^t attains its maximum on ∂D_i^t . Thus we can omit the absolute values of the normal derivatives and obtain

$$\begin{aligned} I &\leq \sup_{D_1^t} (1 - v_2^t) \int_{\partial D_1^t} \partial_\nu v_1^t |\nabla v_1^t|^{p-2} + \sup_{D_2^t} (1 - v_1^t) \int_{\partial D_2^t} \partial_\nu v_2^t |\nabla v_2^t|^{p-2} \\ &= \sup_{D_1^t} (1 - v_2^t) \int_{B_{R_\varepsilon} \setminus D_1^t} \operatorname{div}(v_1 \nabla v_1^t |\nabla v_1^t|^{p-2}) + \sup_{D_2^t} (1 - v_1^t) \int_{B_{R_\varepsilon} \setminus D_2^t} \operatorname{div}(v_2 \nabla v_2^t |\nabla v_2^t|^{p-2}) \\ &= \sup_{D_1^t} (1 - v_2^t) \int_{B_{R_\varepsilon} \setminus D_1^t} |\nabla v_1^t|^p + \sup_{D_2^t} (1 - v_1^t) \int_{B_{R_\varepsilon} \setminus D_2^t} |\nabla v_2^t|^p. \end{aligned}$$

Recall that by Lemma 5.7 [6] there is a generic constant $M > 0$ such that

$$(|\xi|^{p-2} \xi - |\eta|^{p-2} \eta)(\xi - \eta) \geq M \begin{cases} |\xi - \eta|^p & \text{if } p > 2, \\ |\xi - \eta|^2 (|\xi| + |\eta|)^{p-2} & \text{if } 1 < p \leq 2 \end{cases} \quad (21)$$

for all $\xi, \eta \in \mathbb{R}^d$.

First suppose that $p > 2$ then applying inequality (21) to (20) yields

$$I \geq M \int_{B_{R_\varepsilon}} |\nabla v_1^t - \nabla v_2^t|^p.$$

As for the case $1 < p \leq 2$ then from (21) we have

$$I \geq M \int_{B_{R_\varepsilon}} |\nabla v_1^t - \nabla v_2^t|^2 (|\nabla v_1^t| + |\nabla v_2^t|)^{p-2}.$$

But, from Hölder's inequality and (18) we get

$$\begin{aligned}
 & \int_{B_{R_\varepsilon}} |\nabla v_1^t - \nabla v_2^t|^p \\
 &= \int_{B_{R_\varepsilon}} |\nabla v_1^t - \nabla v_2^t|^p (|\nabla v_1^t| + |\nabla v_2^t|)^{\frac{p(p-2)}{2}} (|\nabla v_1^t| + |\nabla v_2^t|)^{-\frac{p(p-2)}{2}} \\
 &\leq \left(\int_{B_{R_\varepsilon}} |\nabla v_1^t - \nabla v_2^t|^2 (|\nabla v_1^t| + |\nabla v_2^t|)^{p-2} \right)^{\frac{p}{2}} \left(\int_{B_{R_\varepsilon}} (|\nabla v_1^t| + |\nabla v_2^t|)^p \right)^{1-\frac{p}{2}} \\
 &\leq \left(\frac{I}{M} \right)^{\frac{p}{2}} (2K)^{1-\frac{p}{2}}.
 \end{aligned} \tag{22}$$

Therefore, there is a tame constant M_0 such that for any $p > 1$ we have

$$\begin{aligned}
 & \int_{B_{R_\varepsilon}} |\nabla v_1^t - \nabla v_2^t|^p \\
 &\leq M_0 \left[\sup_{D_1^t} (1 - v_2^t) \int_{B_{R_\varepsilon} \setminus D_1^t} |\nabla v_1^t|^p + \sup_{D_2^t} (1 - v_1^t) \int_{B_{R_\varepsilon} \setminus D_2^t} |\nabla v_2^t|^p \right]^{\min(1, \frac{p}{2})}.
 \end{aligned}$$

Letting $t \rightarrow 0$ we get

$$\begin{aligned}
 \int_{B_{R_\varepsilon}} |\nabla v_1 - \nabla v_2|^p &\leq \liminf_{t \rightarrow 0} \int_{B_{R_\varepsilon}} |\nabla v_1^t - \nabla v_2^t|^p \\
 &\leq M_1 \liminf_{t \rightarrow 0} \left[\sup_{D_1^t} (1 - v_2^t) + \sup_{D_2^t} (1 - v_1^t) \right]^{\min(1, \frac{p}{2})}.
 \end{aligned} \tag{23}$$

with some tame constant M_1 .

Since $1 - v_i^t$ are nonnegative p -subsolutions in B_{R_ε} , from the weak maximum principle, Theorem 3.9 [6] we obtain

$$\sup_{B_{\sigma r}(z)} (1 - v_i^t) \leq \frac{C}{(1 - \sigma)^{n/p}} \left(\int_{B_r(z)} (1 - v_i^t)^p \right)^{\frac{1}{p}}. \tag{24}$$

Take a finite covering of D_j^t with balls $B_r(z_k^i)$, $z_k^i \in S_i$, $r = 3a_\varepsilon$, $k = 1, \dots, N$. Choose t small enough such that $D_j^t \subset \bigcup_{k=1}^N B_r(z_k^i)$ and applying (24) we obtain for $i, j \in \{1, 2\}$ with $i \neq j$

$$\sup_{D_j^t} (1 - v_i^t) \leq \max_k \sup_{B_r(z_k^i)} (1 - v_i^t) \leq C \max_k \left(\int_{B_{2r}(z_k^i)} (1 - v_i^t)^p \right)^{\frac{1}{p}}.$$

Since $\|v_i^t\|_{W^{1,p}(B_3)} \leq C$ uniformly for all $t > 0$ it follows that $v_i^t \rightarrow v_1$ strongly in $L^p(B_3)$ and v_i is quasi-continuous. In other words, for any positive number θ there is a set E_θ such that $\text{cap}_p E_\theta < \theta$ and v_i is continuous in $B_2 \setminus E_\theta$. Notice that $E_\theta \subset S_1 \cup S_2$ and hence $\mathcal{H}^d(E_\theta) = 0$.

This yields

$$\begin{aligned} \lim_{t \rightarrow 0} \int_{B_r(z_k^i)} (1 - v_i^t)^p &= \int_{B_r(z_k^i)} (1 - v_i)^p = \int_{B_{2r}(z_k^i) \cap E_\theta} (1 - v_i)^p \\ &\quad + \int_{B_{2r}(z_k^i) \setminus E_\theta} (1 - v_i)^p \\ &= \int_{B_{2r}(z_k^i) \setminus E_\theta} (1 - v_i)^p \leq C[\omega_i(6a_\varepsilon)]^p \end{aligned} \quad (25)$$

where $\omega_i(\cdot)$ is the modulus of continuity of v_i on B_3 modulo the set E_θ . Thus

$$\int_{B_{R_\varepsilon}} |\nabla v_1 - \nabla v_2|^p \leq C[\omega_1(6a_\varepsilon) + \omega_2(6a_\varepsilon)]^p \min(1, \frac{p}{2}).$$

Hence (17) is established. Rescaling back and noting that $a_\varepsilon^{d-p} \omega_i(a_\varepsilon) = o(a_\varepsilon^{d-p})$ the result follows. Observe that L^p norm of ∇v_i^t remains uniformly bounded in B_{R_ε} by (18) and hence the moduli of quasi-continuity in, say, B_3 do not depend on the particular choice of Γ_ε^k or the tangent plane P_x^k .

2° We recast the argument above but now for $S_1 = \frac{1}{a_\varepsilon} P_1$, $S_2 = \frac{1}{a_\varepsilon} P_2$. Squaring the inequality $|v_1 - v_2| \leq \delta$ we get that $2 \sin \frac{\beta}{2} \leq \delta$ where β is the angle between P_1 and P_2 . Since δ now measures the deviation of v_1^t from 1 on D_2^t , (resp. v_2^t on D_1^t) we conclude that the corresponding moduli of continuity of the limits v_1, v_2 (as $t \rightarrow 0$) modulo a set $E_\theta \subset S_1 \cup S_2$ with small p -capacity depend on δ , i.e.

$$\int_{B_r(z_k^i)} (1 - v_i)^p \leq C[\omega_i(12\delta)]^p \quad (26)$$

where $B_r(z_k^i)$ provide a covering of D_i^t as above but now, say, $r = 6\delta$. Hence we can take $c_\delta = C(\omega_1(12\delta) + \omega_2(12\delta))$. \square

5 Proof of Theorem 3

We now formulate our result on the local approximation of total capacity (say in Q') by tangent planes of Γ and prove (16).

Lemma 1 Fix a cube $Q' \subset \mathbb{R}^{d-1}$ such that if $x = (x', x_d)$ and $y = (y', y_d)$ belong to Γ and $x', y' \in Q'$, then the normals v_x, v_y of Γ at x and y satisfy $|v_x - v_y| \leq \delta$. Then for any $x = (x', x_d) \in \Gamma$ with $x' \in Q'$, there holds

$$\lim_{\varepsilon \rightarrow 0} \sum_{k \in \mathbb{Z}^n : k' \in \varepsilon^{-1} Q'} \int_{B_\varepsilon^k} |\nabla w_\varepsilon^k|^p dx = [\text{cap}_{p, v_x}(T) + O(C_\delta)] \mathcal{H}^{d-1}(\Gamma_{Q'}),$$

where $\lim_{\delta \rightarrow 0} C_\delta = 0$ and $\Gamma_{Q'} = \{x \in \Gamma : x' \in Q'\}$.

Proof Fix $x \in \Gamma_{Q'}$ and let P be the plane $\{y : y \cdot v_x = 0\}$, where v_x is the normal of Γ at x . Suppose $k = (k', k_d) \in \mathbb{Z}^d$, $\varepsilon k' \in Q'$ and let P_{x^k} be the tangent plane to Γ at $x^k = (\varepsilon k', g(\varepsilon k'))$. Then Theorem 2 **1°** tells us that

$$\text{cap}_p(\Gamma_\varepsilon^k) = \text{cap}_p(P_{x^k} \cap T_\varepsilon^k) + o(a_\varepsilon^{d-p}).$$

If we set $P_\varepsilon^k = P + (-\varepsilon k', g(\varepsilon k'))$, then P_ε^k will intersect the point $(\varepsilon k', g(\varepsilon k'))$. By assumption, $|\nu_x - \nu_{x^k}| \leq \delta$, so

$$\text{cap}_p(P_\varepsilon^k \cap T_\varepsilon^k) = \text{cap}_p(P_{x^k} \cap T_\varepsilon^k) + O(c_\delta a_\varepsilon^{d-p}),$$

by Theorem 2.2°. This gives $\text{cap}_p(\Gamma_\varepsilon^k) = \text{cap}_p(P_\varepsilon^k \cap T_\varepsilon^k) + O(c_\delta a_\varepsilon^{d-p})$. Since, by Theorem 1, the sequence $\{\varepsilon^{-1}g(\varepsilon k')\}_{k' \in \varepsilon^{-1}Q'}$ is uniformly distributed mod 1 with discrepancy of order $\varepsilon^{1/3}$, the rescaled planes $\varepsilon^{-1}P_\varepsilon^k$ have the same distribution mod 1, i.e. they are translates of P and the translates have the same distribution. Using the proof of Lemma 4 of [4], we conclude that

$$\lim_{\varepsilon \rightarrow 0} \sum_{k \in \mathbb{Z}^n: k' \in \varepsilon^{-1}Q'} \text{cap}_p(\{P_\varepsilon^k\} \cap T_\varepsilon^k) = \text{cap}_{p, \nu_x}(T) \mathcal{H}^{d-1}(P_{Q'}),$$

where $P_{Q'} = \{x \in P: x' \in Q'\}$. Since we know that $\int_{B_\varepsilon^k} |\nabla w_\varepsilon^k|^p dx = \text{cap}_p(\Gamma_\varepsilon^k) + o(a_\varepsilon^{d-p})$, the result follows from the fact that $\mathcal{H}^{d-1}(\Gamma_{Q'}) = (1 + O(c_\delta)) \mathcal{H}^{d-1}(P_{Q'})$. \square

Lemma 2

$$\lim_{\varepsilon \rightarrow 0} \int_Q |\nabla w_\varepsilon|^p dx = \int_{\Gamma \cap Q} \text{cap}_{p, \nu_x}(T) d\mathcal{H}^{d-1}.$$

Proof The claim follows by decomposing the set $\{x' \in \mathbb{R}^{d-1}: (x', g(x')) \in \Gamma \cap Q\}$ into disjoint cubes $\{Q'_j\}$ that satisfy the hypothesis of Lemma 1. Since Γ is C^2 , we can find a finite number of disjoint cubes $\{Q_j\}_{j=1}^{N(\delta)}$, such that $\mathcal{H}^{d-1}(\Gamma \cap Q \setminus \bigcup_j Q_j \cap \Gamma) = 0$ and Q'_j is as in Lemma 1. For all $x \in \Gamma \cap Q_j$ we have $x = (x', g(x))$ for $x' \in Q'_j$, after interchanging coordinate axes if necessary. Thus

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_Q |\nabla w_\varepsilon|^p dx &= \sum_j \lim_{\varepsilon \rightarrow 0} \sum_{k \in \mathbb{Z}^n: k' \in \varepsilon^{-1}Q'_j} \int_{B_\varepsilon^k} |\nabla w_\varepsilon^k|^p dx \\ &= \sum_{x^j \in Q'_j} [\text{cap}_{p, \nu_{x^j}}(T) + O(C_\delta)] \mathcal{H}^{d-1}(\Gamma_{Q'_j}) \\ &= \int_{\Gamma \cap Q} \text{cap}_{p, \nu(x)}(T) d\mathcal{H}^{d-1} + O(C_\delta), \end{aligned}$$

where in the last step we used that $\text{cap}_{p, \nu(x)}(T) = \text{cap}_{p, \nu_{x^j}}(T) + O(C_\delta)$ for all $x \in \Gamma_{Q'_j}$, by Lemma 1. Sending $\delta \rightarrow 0$ proves the lemma. \square

Having established Lemma 2, the rest of the proof of **H₂** and **H₃** is carried out precisely as in [4], with Lemma 2 above replacing Lemma 4 in [4]. The proof of Theorem 3 from **H₁**–**H₃** is given in section 4 of [4] when Γ is a hyper plane, and remains the same for the present case when Γ is a convex surface.

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